Optimization of Multistage Processes

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The maximum range of a rocket vehicle is discussed applying the variational theory of optimum multistage processes. The vector thrust may be controlled in magnitude (within bounds) and in direction, and the two-stage vehicle considered is assumed flying in a vacuum. The necessary staging conditions for an extremal are derived from a determinant form, which relates the influence functions (Lagrange multipliers), the Hamiltonian, and the staging constraints. An important fundamental continuity condition, which permits analyzing the originally discontinuous problem in terms of a continuous one, is derived. This permits a complete solution regarding type, number, and sequence of subarcs forming the extremal. It is shown that on the optimum trajectory the vector velocity coincides with the vector thrust at the initial point and that it is perpendicular to the vector thrust at the final point. The direction of the vector thrust is constant throughout stages. No variable thrust subarc, or coasting subarc, is admissible between stages. These results are valid for N stages. Results previously obtained for single-stage vehicles are therefore extended to multistage vehicles.

Nomenclature

c = average velocity of the gases at the exit section of the nozzle, fps
F = fundamental function in terms of the canonical variables
g = acceleration of gravity, ft/sec^2
H = Hamiltonian
k = const
m = mass of the vehicle, lb-sec^2/ft
q = generalized coordinate
t = time, sec
u = horizontal component of the vector velocity, fps
v = vertical component of the vector velocity, fps
x = range, ft
y = altitude, ft
α = angle between the vector velocity and the horizontal
β = mass flow of the rocket engine, lb-sec/ft
$\delta(\ )$ = variation of ( )
θ = implicit form of the terminal conditions
λ_0 = constant Lagrange multiplier
Λ = Euler-Lagrange sum
μ = variable Lagrange multiplier
ν = canonical influence function
Π = function to be minimized
φ = angle between the vector thrust and the horizontal axis
ψ = implicit form of the equations of motion

Superscripts

$(\ )^*$ = quantity evaluated at staging time
$(\ )'$ = total derivative of ( ) with respect to time

Subscripts

I = quantity evaluated at the initial time
F = quantity evaluated at the final time
c = quantity evaluated at a corner time

Introduction

PREVIOUS papers[1–6] have covered the optimization process for single-stage vehicles. Variational studies of the optimum trajectory of a multistage vehicle (i.e., the state variable mass experiences step discontinuities or finite jumps at certain times within the interval of integration) have been presented in Refs. 7 and 8. In Ref. 8, necessary and sufficient conditions for an optimum trajectory of a multistage rocket are given using the Green's theorem approach. The latter technique is applicable here[8] owing to the low dimensionality of the case treated. In Ref. 7 it was shown that the necessary condition obtained[8] may be completely derived using the theory of optimum multistage processes. The latter may be applied to treat multistage cases of high dimensionality (i.e., three-dimensional arbitrary trajectories). This theory is applied in the present paper. It is shown that the theory permits the determination of optimum staging and the number, type, and sequence of subarcs in the trajectory of an n-stage rocket.

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Variational Formulation: Legendre Transformation of the Problem into Canonical Form

The problem proposed is that of finding, in the class of possible trajectories that satisfy the following equations of motion of a mass-point rocket subject to a uniform gravitational field in a vacuum

$$\psi_1 \equiv x' - u = 0 \tag{1}$$

$$\psi_2 \equiv y' - v = 0 \tag{2}$$

$$\psi_3 \equiv u' - (\beta c/m)\cos\varphi = 0 \tag{3}$$

$$\psi_4 \equiv v' + g - (\beta c/m)\sin\varphi = 0 \tag{4}$$

$$\psi_5 \equiv m' + \beta = 0 \tag{5}$$

that one trajectory that minimizes the terminal range function

$$\Pi = -x_F \tag{6}$$

subject to the terminal constraints

$$\left.\begin{aligned} \theta_1 &\equiv x_I = 0 \\ \theta_2 &\equiv t_I = 0 \\ \theta_3 &\equiv (u_I^2 + v_I^2)^{1/2} - k_1 = 0 \\ \theta_4 &\equiv y_I - k_2 = 0 \\ \theta_5 &\equiv m_I - k_3 = 0 \\ \theta_6 &\equiv y_F - k_4 = 0 \\ \theta_7 &\equiv m_F - k_5 = 0 \end{aligned}\right\} \tag{7}$$

The state variables x, y, u, v are of class D',[9] in the interval $t_I \leqq t \leqq t_F$, whereas the variable $m(t)$ may be of class D' only in the subintervals $t_I \leqq t \leqq t_1^*$ and $t_1^* \leqq t \leqq t_2^*$, and presents finite discontinuities at the staging time t_1^* and t_2^* at which the equations

$$\theta_8 \equiv m(t_1^* - 0) - m(t_1^* + 0) - k_6 = 0 \tag{8}$$

$$\theta_9 = m(t_1^* - 0) - k_7 = 0 \tag{9}$$

$$\theta_{10} \equiv m(t_2^* - 0) - m(t_2^* + 0) - k_8 = 0 \tag{10}$$

$$\theta_{11} = m(t_2^* - 0) - k_9 = 0 \tag{11}$$

are satisfied. The coordinate system, angles, and forces are shown in Fig. 1. The problem has four degrees of freedom associated with the two control functions $\varphi(t)$, $\beta(t)$ and with the staging times t_1^* and t_2^*. We desire to optimize these freedoms so as to attain the maximum range. It is assumed that in each stage the thrust may be regulated, if desired, and that its magnitude is bounded. Thus, since the velocity of the gases at the exit of the rocket nozzle is assumed constant, $\beta_{1_{\min}} \leqq \beta_1 \leqq \beta_{1_{\max}}$ and $\beta_{2_{\min}} \leqq \beta_2 \leqq \beta_{2_{\max}}$, where the subindexes 1 and 2 indicate first and second stages, respectively, and $\beta_{1_{\min}} = \beta_{2_{\min}} = 0$. The control variable φ is unbounded. The weights of propellant and casing of each stage and payload are known.

The state variables will be now denoted by the generalized coordinate q_j, $j = 1, \ldots, 5$. Thus $q_1 = x$, $q_2 = y$, $q_3 = u$, $q_4 = v$, and $q_5 = m$. The canonical variables $(t, q_j, \beta, \varphi, \nu_j;$ $j = 1, \ldots, 5)$, which are related to the variables $(t, q_j, q_j', \beta, \varphi, \mu_i;$ $i = 1, \ldots, 5)$ by the equations[13,15]

$$\nu_j = \Lambda_{q_j'}(t, q_j, q_j', \beta, \varphi, \mu_i) \qquad \psi_i(t, q_j, q_j', \beta, \varphi) = 0 \tag{12}$$

are now introduced. Here ν_j is a set of variable canonical influence functions, $\Lambda = \mu_i\psi_i$ $(i = 1, \ldots, 5)$ is the Euler-Lagrange sum, and μ_i is a set of variable Lagrange multipliers.[14,16] The variational problem may be formulated in canonical form[10] by introducing the fundamental function

$$F(t, q_j, \beta, \varphi, \nu_j) = \nu_j(dq_j/dt) - H \tag{13}$$

where H is the Hamiltonian

$$H(t, q_j, \beta, \varphi, \nu_j) = \nu_j q_j' - \Lambda \tag{14}$$

The formulation of the variational problem in canonical form thus involves the Legendre transformation implied by Eqs. (12) and (14), i.e., the introduction of the canonical variables ν_j and the Hamiltonian H.

From the first variation problem[7,10] it may be obtained that the necessary conditions for an extremal in canonical form are that the Euler-Lagrange sum satisfy $\nu_j' + H_{q_j} = 0$, $q_j' - H_{\nu_j} = 0$, and $H' - H_t = 0$, on every subarc composing the extremal, with a set of nonsimultaneously vanishing canonical variables ν_j, such that the matrix

$$\begin{vmatrix} \lambda_0 \dfrac{\partial \Pi}{\partial q_{jI}} - \nu_{jI} & \lambda_0 \dfrac{\partial \Pi}{\partial q_{jF}} + \nu_{jF} & \lambda_0 \dfrac{\partial \Pi}{\partial t_I} + H_I & \times \\ & & & \lambda_0 \dfrac{\partial \Pi}{\partial t_F} - H_F \\ \dfrac{\partial \theta_\rho}{\partial q_{jI}} & \dfrac{\partial \theta_\rho}{\partial q_{jF}} & \dfrac{\partial \theta_\rho}{\partial t_I} & \dfrac{\partial \theta_\rho}{\partial t_F} \end{vmatrix} \tag{15}$$

$(\rho = 1, \ldots, r = 7,$ $\lambda_0 =$ constant multiplier) is of rank $R < r + 1 = 8$, satisfying at junction of subarcs (corners) the Erdmann-Weierstrass vertex continuity conditions $\nu_j(t_c - 0) = \nu_j(t_c + 0)$ $(j = 1, \ldots, 5)$ and $H(t_c - 0) = H(t_c + 0)$. On each subarc with unrestricted control variations $\delta\beta \gtreqless 0$ and $\delta\varphi \gtreqless 0$ or restricted control variations $\delta\beta \geqq 0$ and $\delta\beta \leqq 0$ (since only β is assumed bounded), the following conditions pertain:

$$\partial H/\partial\varphi = 0, \quad \delta\varphi \gtreqless 0; \quad \partial H/\partial\beta = 0, \quad \delta\beta \gtreqless 0 \tag{16a}$$

$$\frac{\partial H}{\partial \beta} \leqq 0, \quad \delta\beta \geqq 0; \quad \text{and} \quad \frac{\partial H}{\partial \beta} \geqq 0, \quad \delta\beta \leqq 0 \tag{16b}$$

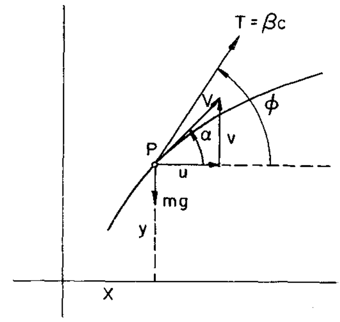

Fig. 1 Coordinate system, angles, and forces.

At the staging times t_1^* and t_2^* (times at which $q_5 = m$ experiences a finite discontinuity Δm), the staging conditions are

$$\begin{vmatrix} \nu_5(t_1^* - 0) & -\nu_5(t_1^* + 0) & H(t_1^* - 0) - H(t_1^* + 0) \\ \dfrac{\partial \theta_8}{\partial m(t_1^* - 0)} & \dfrac{\partial \theta_8}{\partial m(t_1^* + 0)} & \dfrac{\partial \theta_8}{\partial t_1^*} \\ \dfrac{\partial \theta_9}{\partial m(t_1^* - 0)} & \dfrac{\partial \theta_9}{\partial m(t_1^* + 0)} & \dfrac{\partial \theta_9}{\partial t_1^*} \end{vmatrix} = 0 \tag{17}$$

$$\begin{vmatrix} \nu_5(t_2^* - 0) & -\nu_5(t_2^* + 0) & H(t_2^* - 0) - H(t_2^* + 0) \\ \dfrac{\partial \theta_{10}}{\partial m(t_2^* - 0)} & \dfrac{\partial \theta_{10}}{\partial m(t_2^* + 0)} & \dfrac{\partial \theta_{10}}{\partial t_2^*} \\ \dfrac{\partial \theta_{11}}{\partial m(t_2^* - 0)} & \dfrac{\partial \theta_{11}}{\partial m(t_2^* + 0)} & \dfrac{\partial \theta_{11}}{\partial t_2^*} \end{vmatrix} = 0 \tag{18}$$

$$\nu_k(t_1^* - 0) - \nu_k(t_1^* + 0) = 0 \qquad k = 1, \ldots, 4 \tag{19}$$

$$\nu_k(t_2^* - 0) - \nu_k(t_2^* + 0) = 0 \qquad k = 1, \ldots, 4 \tag{20}$$

Equation (15) is the matrix form of the transversality condition, and since normal, nonsingular extremals are assumed, $\lambda_0 = 1$. If the mass before staging $m(t^* - 0)$ were not specified, Eqs. (17) and (18) would reduce to matrix forms.[7] The necessary conditions for staging determine the continuity conditions on the ν_5 canonical variable and on the Hamiltonian H at points of mass discontinuity.

Extremal Solution and Staging Conditions

The Hamiltonian is expressed by

$$H = \nu_1 u + \nu_2 v - \nu_4 g + (\nu_3 \cos\varphi + \nu_4 \sin\varphi)\beta c/m - \nu_5\beta \tag{21}$$

Note that using the canonical formulation of the variational problem the Weierstrass necessary condition readily leads to the maximality principle

$$W \equiv \Delta\Lambda - \Lambda_{q_j'}\Delta q_j' = -\Delta H \geqq 0 \tag{22}$$

Therefore $\Delta H \leqq 0$. The canonical equations of the extremals are

$$\nu_1' = 0 \qquad \nu_1 = k_{10} \tag{23}$$

$$\nu_2' = 0 \qquad \nu_2 = k_{11} \tag{24}$$

$$\nu_3' = -\nu_1 \qquad \nu_3 = \nu_{3I} - k_{10}t = \nu_{3F} - k_{10}(t - t_F) \tag{25}$$

$$\nu_4' = -\nu_2 \qquad \nu_4 = \nu_{4I} - k_{11}t = \nu_{4F} - k_{11}(t - t_F) \tag{26}$$

$$\nu_5' = (\nu_3 \cos\varphi + \nu_4 \sin\varphi)\beta c/m^2 \tag{27}$$

$$H_t = 0 \qquad H' = 0 \qquad H(t) = \text{const}$$
$$t_I \leqq t \leqq t_1^*,\ t_1^* \leqq t \leqq t_2^*,\ t_2^* \leqq t \leqq t_F \tag{28}$$

The multipliers ν_1, ν_2, ν_3, and ν_4 are continuous in $t_I \leqq t \leqq t_F$ owing to the vertex and staging continuity conditions.

Equations (16) lead to the optimum control equations

$$H_\varphi = \nu_3 \sin\varphi - \nu_4 \cos\varphi = 0 \qquad \delta\varphi \gtreqless 0 \tag{29}$$

$$H_\beta = (\nu_3 \cos\varphi + \nu_4 \sin\varphi)c/m - \nu_5 = 0$$

$$0 \leqq \beta \leqq \beta_{\max} \tag{30}$$

$$H_\beta \leqq 0 \qquad \beta = 0 \tag{31}$$

$$H_\beta \geqq 0 \qquad \beta = \beta_{\max} \tag{32}$$

From Eq. (29) it follows that

$$\tan\varphi = \frac{\nu_4}{\nu_3} = \frac{\nu_{4I} - k_{11}t}{\nu_{3I} - k_{10}t} = \frac{\nu_{4F} - k_{11}(t - t_F)}{\nu_{3F} - k_{10}(t - t_F)} \tag{33}$$

From the expanded form of the transversality condition and the terminal constraints, the following natural boundary conditions may be derived:

$$(\nu_{1F} - 1)dx_F = 0 \qquad dx_F \neq 0 \qquad \nu_1(t) = k_{10} = 1 \qquad t_I \leqq t \leqq t_F \tag{34}$$

$$\nu_{3F}du_F = 0 \qquad du_F \neq 0 \qquad \nu_{3F} = 0 \tag{35}$$

$$\nu_{4F}dv_F = 0 \qquad dv_F \neq 0 \qquad \nu_{4F} = 0 \tag{36}$$

$$H_F dt_F = 0 \qquad dt_F \neq 0 \qquad H(t) = H_F = 0 \qquad t_2^* \leqq t \leqq t_F \tag{37}$$

$$\left.\begin{aligned} \nu_{3I}du_I + \nu_{4I}dv_I &= 0 \\ u_I du_I + v_I dv_I &= 0 \end{aligned}\right\} \nu_{3I} = k_{12}u_I \qquad \nu_{4I} = k_{12}v_I \tag{38}$$

From Eqs. (25, 26, 35, and 36)

$$k_{11}/k_{10} = \nu_{4I}/\nu_{3I} \tag{39}$$

and therefore, the bilinear function in Eq. (33) together with Eqs. (38) and (39) lead to

$$\tan\varphi = k_{11}/k_{10} = \nu_{4I}/\nu_{3I} = (v/u)_I = \text{const} \tag{40}$$

Equation (40) indicates that the thrust vector is tangent to the trajectory at the initial point and that its direction remains constant along every powered arc[1–6, 11, 12] on any stage.

It is assumed that at the initial point $0 < \tan^{-1}(v/u)_I < \pi/2$; thus, since $k_{10} = 1$, then $k_{11} > 0$.

Nonexistence of β-Variable Subarcs

The total derivative with respect to time of Eq. (30) gives

$$k_{10} \cos\varphi + k_{11} \sin\varphi = 0 \tag{41}$$

and then

$$\tan\varphi = -k_{10}/k_{11} = \text{const} \tag{42}$$

Equation (42) contradicts Eq. (40), which is associated with the optimum φ control along any extremal subarc. Thus, it is concluded that no variable thrust subarc may form any part of the extremal arc.[1–6, 11, 12] Only β-constant subarcs may form the extremal arc.

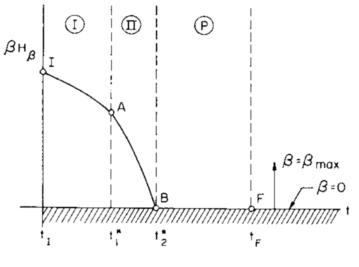

Fig. 2 Vehicle considered and fundamental function vs time

Staging and Sequence of Subarcs

In previous paragraphs we have determined the type of subarcs that may form the extremal arc. Applying the necessary staging conditions presented, we will show that one can determine the proper sequence and number of subarcs that may form the extremal: in other words, the optimum $\beta(t)$ program (since φ is constant) and the optimum staging times t_1^* and t_2^*.

To this extent, from Eqs. (17) and (18) is derived

$$\begin{vmatrix} \nu_5(t^* - 0) & -\nu_5(t^* + 0) & H(t^* - 0) - H(t^* + 0) \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 \tag{43}$$

which, for $t^* = t_1^*$ and $t^* = t_2^*$, gives the necessary staging conditions

$$H(t_1^* - 0) = H(t_1^* + 0) \qquad H(t_2^* - 0) = H(t_2^* + 0) \tag{44}$$

Thus, for the problem proposed, the Hamiltonian is continuous along the extremal. Consequently, Eqs. (28) and (37) lead to

$$H(t) = 0 \qquad t_I \leqq t \leqq t_F \tag{45}$$

From these considerations, the fundamental continuity condition

$$\beta H_\beta\Big|_{t-0}^{t+0} = 0 \qquad t_I \leqq t \leqq t_F \tag{46}$$

is derived. Also, from Eqs. (31) and (32) it follows that

$$\beta H_\beta \geqq 0, \quad \beta = \beta_{\max}; \quad \text{and} \quad \beta H_\beta = 0, \quad \beta = \beta_{\min} = 0 \tag{47}$$

In addition, the total derivative with respect to time of the fundamental function βH_β leads to

$$d(\beta H_\beta)/dt = -(k_{10} \cos\varphi + k_{11} \sin\varphi)\beta c/m \tag{48}$$

and therefore

$$d(\beta H_\beta)/dt < 0, \quad \beta = \beta_{\max}; \quad \text{and} \quad d(\beta H_\beta)/dt = 0, \quad \beta = 0 \tag{49}$$

Equations (47) and (49) uniquely determine the type, sequence, and number of subarcs forming the extremal arc. The extremal can only be formed by three subarcs and, in general, for an n-stage rocket vehicle, by $n + 1$ subarcs. The first and second stages must be burned at full-thrust, one after the other, without any intermediate coasting subarc.

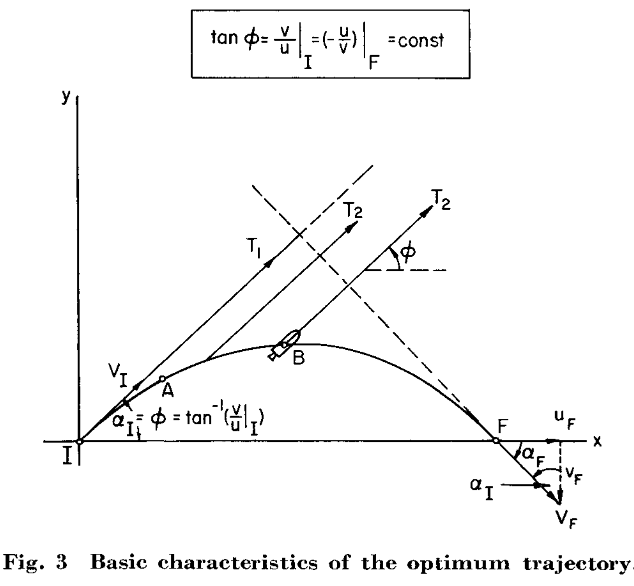

Fig. 3 Basic characteristics of the optimum trajectory.

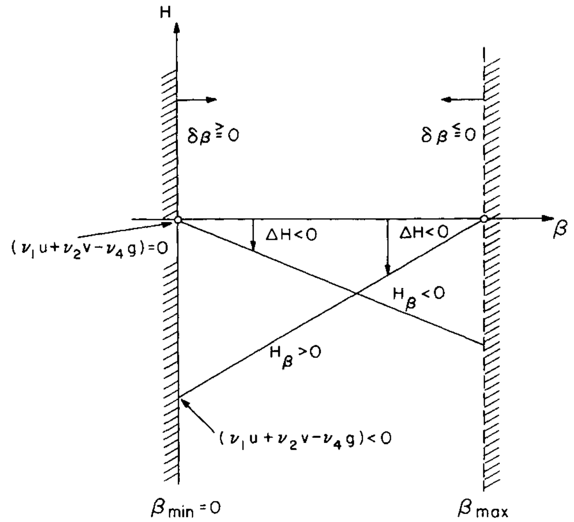

Fig. 4 Maximality principle (Weierstrass condition) for restricted control variations.

The βH_β function along the extremal, for the two-stage rocket vehicle assumed, is schematically shown in Fig. 2. Since $\beta = 0$ along the final subarc,

$$k_{10}u + k_{11}v|_F = 0 \qquad k_{11}/k_{10} = -(u/v)_F \tag{50}$$

Equations (40) and (50) imply that

$$\tan\varphi = (v/u)_I = -(u/v)_F = \text{const} \tag{51}$$

which means that the optimum trajectory is such that the vector thrust is tangent to the trajectory at the initial point and perpendicular to the vector velocity at the final point (as has been shown for single stage vehicles[2]). This condition, assuming $k_2 = k_4 = 0$ in Eq. (7), is graphically represented in Fig. 3.

The maximality principle for the case considered is represented in Fig. 4. Finally, Fig. 5 shows a graphical interpretation of the necessary conditions for staging. Thus, the optimum burning program $\beta(t)$ and thrust orientation angle $\varphi(t)$ have been obtained as well as the optimum staging times, i.e.,

$$t_1^* = m_{p1}/\beta_{1_{max}} \quad \text{and} \quad t_2^* = t_1^* + m_{p2}/\beta_{2_{max}}$$

where m_p is the propellant mass.

Conclusions

It has been shown that the maximum range trajectory of a two-stage rocket vehicle flying in a vacuum is composed of three subarcs. The first two correspond to full-thrust burning of the stages and the final one corresponds to the coasting flight of the payload. No variable-thrust subarc is admissible, nor is any coasting subarc between stages. The thrust direction is constant along the powered subarcs. The extremal arc is such that the vector thrust is tangent to the trajectory at the initial point and perpendicular to the vector velocity at the final point. These conditions, previously detected for single-stage vehicles, have been here shown to apply to multistage vehicles.

The staging conditions presented permit the determination of a fundamental function that allows us to detect the type, sequence, and number of subarcs forming the extremal arc. The conclusions also apply to n-stage vehicles.

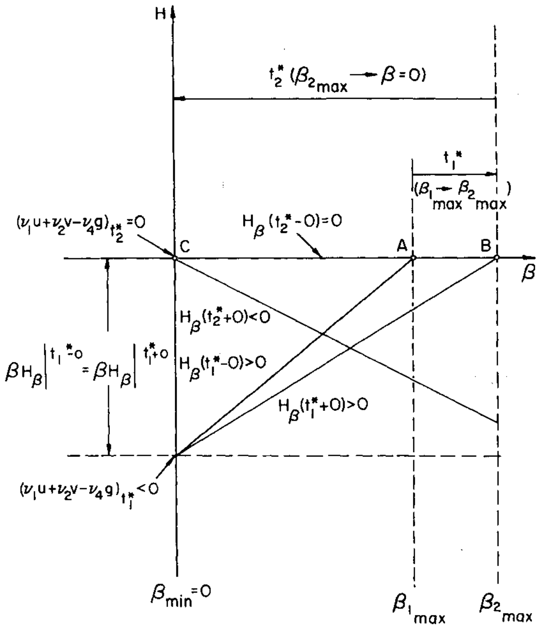

Fig. 5 Graphical interpretation of the continuity conditions at the staging times t_1* and t_2*.

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